Separable programming for aggregate production planning — A high-order cost case

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Many production managers are faced with the problem of planning production, inventory and work-force under the constraint of limited resources to meet a seasonal demand. Considerable research has been done on this planning problem and various planning models have been introduced. In those cases where linearity of the cost functions of an undertaking may reasonably be assumed, an ordinary linear programming model suffices. In many cases, however, this simple linear approach to certain essentially non-linear cost functions is unacceptable owing to the gross approximation made.

Separable programming (SEP) is introduced as a solution methodology to this aggregate production planning problem in a complex, high-order cost structure case. The cost structure was used by Goodman for the application of goal programming (GP) in this field. The Goodman GP model makes provision for positive or negative slack for the production level, work-force level and inventory level with penalty costs for these slack-deviations. Goodman also made use of a 'sectioning search' model for this high-order cost case to serve as a measure for his GP model. A comparison is made between the results of these three approaches. SEP offered an improvement of more than 4% in total cost in comparison with the sectioning search model, and performs 26% better than the GP model.

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Baie produksiebestuurders word gekonfronteer met die probleem van die beplanning van produksiehoeveelheid, voorraadvlak en arbeidsmag met inagneming van die beperkte bronne tot die onderneming se beskikking vir die bevrediging van 'n seisoenale aanvraag. Heelwat navorsing is al oor hierdie beplanningsprobleem gedoen en 'n verskeidenheid wiskundige en ander modelle is getoets. In die gevalle waar lineêre kostefunksies by 'n onderneming met 'n groot mate van sekerheid aanvaar kan word, kan van gewone lineêre programmeringsmodelle gebruik gemaak word. In baie ander gevalle is dié aanname van 'n lineêre kostestruktuur egter onaanvaarbaar weens growwe aannames wat gemaak word.

Skeibare programmering (SEP) word voorgestel as 'n oplossingsmetodiek vir die taktiese produksiebeplanningsprobleem in 'n komplekse, hoë-orde, kostestruktuurgeval. Hierdie kostestruktuur is deur Goodman gebruik vir die toepassing van doelwit-programmering (GP) in dié gebied. Die Goodman doelwit-programmeringsmodel maak voorsiening vir positiewe en negatiewe afwykings vir die produksievlak, arbeidsmagvlak en voorraadvlak met boete-koste vir die afwykings. Goodman gebruik ook 'n 'verdeling-soek'-model, vir hierdie hoë-orde koste-geval, vir vergelykingsdoeleindes teen sy GP-model. 'n Vergelyking word gemaak tussen die resultate van hierdie drie benaderings. Die SEP-model toon 'n verbetering van 4% op die totale koste van die verdeling-soek-model en presteer 26% beter as die GP-model.

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The aggregate production planning problem, in its simplest form, may be stated as follows:

To develop the lowest cost production plan given a fluctuating demand pattern and limited production resources. Various solution methodologies to solve this optimization problem have been suggested. Holt, Modigliani, Muth and Simon[1,2] developed the Linear Decision Rule (LDR) approach for assumed quadratic cost structures. Their published application of LDR at a paint factory served as a yardstick for many different models. Hanssman and Hess[3] used the LDR model as a basis for the development of their programming model because . . . 'it appears, however, that in the majority of practical applications and theoretical models the cost functions are assumed to be linear'.

Based on this approach Goodman[4,5] presented an alternative linearization method and used goal programming (GP) to solve the paint factory problem. In comparison with linear programming (LP) and LDR, goal programming gave good results for the quadratic cost function case. However, in the case of the higher-order cost function used by Goodman, the GP approach failed in comparison with a computer search method (sectioning search).

In this paper the use of separable programming (SEP) for the case of high-order cost functions is shown to give excellent results. The advantage of this method lies in the fact that ordinary linear programming algorithms can be used to solve the model. To develop his fourth-order cost function Goodman made use of a hypothetical real world set of data, reproduced in Table 1. SEP is also applied to this set of data to test its ability to fit the real world situation.

Goal programming model

The Goodman GP model was developed to fit the hypothetical historical cost data-set shown in Table 1. In this table only the absolute values of changes are shown. It is assumed that the costs are symmetrical about the zero cost point of each variable.

The cost model is constructed as follows. Define:

P_t = Production rate in period t
D_t = Demand in period t
I_t = Inventory level at the end of period t
W_t = Work-force level in period t

By fitting curvilinear segments to a set of the hypothetical cost data the following cost functions were obtained:

$340\ W_t$	Regular payroll.
$0{,}2\ (P_t - 6\ W_t)^4$	Overtime and idletime
$64(W_t - W_{t-1})^4$	Hiring and layoff.
$0{,}1(P_t - P_{t-1})^4$	Production level increase and decrease.
$0{,}1(I_t - 320)^4$	Inventory and shortages.

These cost functions lead to the following total cost model. Minimize the total cost:

$$\sum_{t=1}^{n} [340\ W_t + 0{,}2(P_t - 6\ W_t)^4 + 64(W_t - W_{t-1})^4 + 0{,}1(P_t - P_{t-1})^4 + 0{,}1(I_t - 320)^4]$$

Subject to:

$$I_t = I_{t-1} + P_t - D_t$$

$$P_t \geqslant 0$$

$$W_t \geqslant 0$$

for $t = 1,2,3 \ldots . n$

Goodman solved this model with goal programming. It is based upon the notion that each of the fourth-order cost terms becomes zero when the expression inside the parentheses is zero. Minimization of each cost term is regarded as a goal and is formulated as a constraint. It is necessary to allow positive and negative slack in these constraints because it is not possible to minimize simultaneously all the cost terms while at the same time satisfying the demand requirements. The resulting goal constraints can be expressed as:

$$P_t - 6W_t + Q_t^+ - Q_t^- = 0$$

$$W_t - W_{t-1} + R_t^+ - R_t^- = 0$$

$$P_t - P_{t-1} + S_t^+ - S_t^- = 0$$

$$I_t - 320 + T_t^+ - T_t^- = 0$$

$$I_{t-1} + P_t - D_t = I_t$$

$$P_t \geqslant 0$$

$$W_t \geqslant 0$$

$$t = 1,2 \ldots . n,$$

where Q, R, S and T are slack variables.

Positive coefficients are assigned to the slack variables in the objective function. The effect is to penalize deviations from the desired goals. The objective function is given by:

$$\text{Min} \sum_{t=1}^{n} [340\ W_t + C_1Q_t^+ + C_1Q_t^- + C_2R_t^+ + C_2R_t^- + C_3S_t^+ + C_3S_t^- + C_4T_t^+ + C_4T_t^-]$$

The coefficients C_1, C_2, C_3 and C_4 must be selected to give good cost approximations of the cost terms that they represent. This is done by approximating the cost functions by linear segments. The slopes of these linear segments give the desired cost coefficients. (See Figure 1.) The slopes are set so that area A is equivalent to area B, for example:

$$C_2 = \frac{y}{\max|W_t - W_{t-1}|}\ .$$

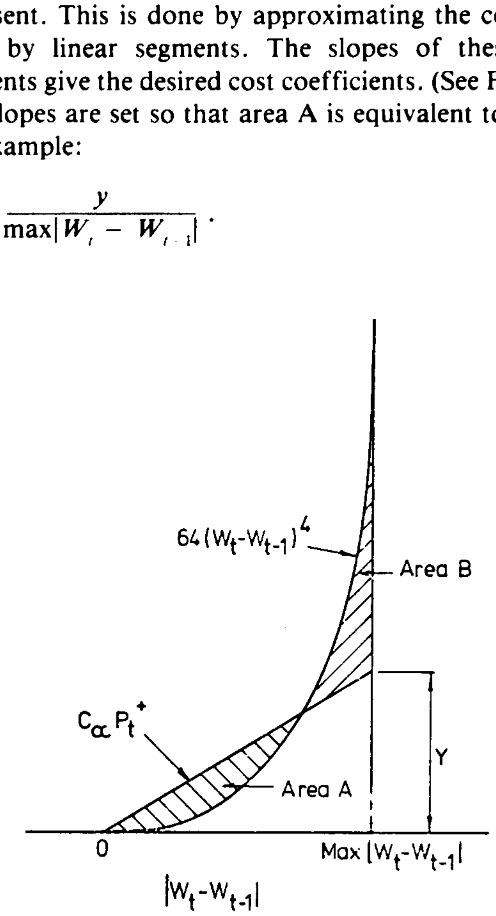

Figure 1 Approximation of a cost function by a linear segment (Goodman).

Separable programming formulation

In separable programming the same principle is applied as in goal programming. The main difference is that the cost terms are now approximated by several linear segments. (See Figure 2.)

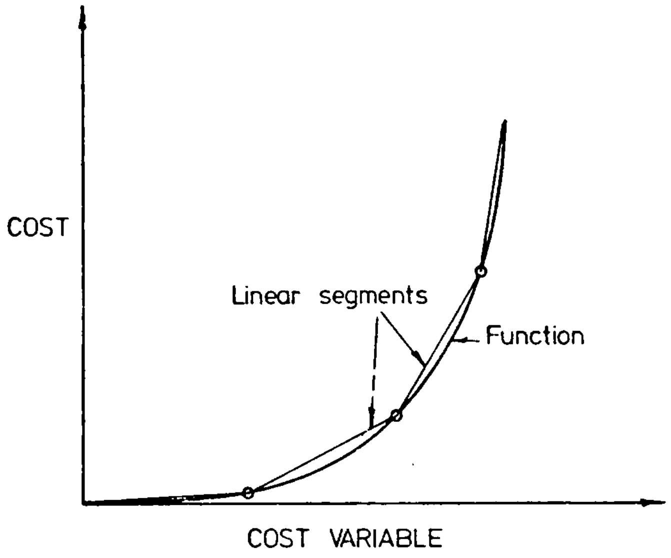

Figure 2 Approximation of high-order cost function by several linear segments.

The following model can thus be formulated. Define:

OT = Overtime
OA = Idle time
HT = Increase in work-force (Number of people)
HA = Decrease in work-force (Number of people)
PT = Increase in production rate
PA = Decrease in production rate
VT = Inventory
VA = Shortages
W = Normal work-force
P = Production rate
I = Net inventory level
D = Demand
t = Time period
T = Number of time periods (Planning horizon)
b, c, d, e, f, g, u, s = Constants from piecewise approximations for variables
B, C, D, E, F, G, U, S = Cost constants from piecewise approximation
i, j, k, l, m, n, q, r = Inter-subscripts determining the sequence of the piecewise segments.
G = Number of piecewise segments required (kept constant to simplify the formulation)
a = Regular work-force cost coefficient
Z = Ratio of production to work-force (productivity constant).

The separable programming model can be expressed as: Minimize the objective function:

$$\sum_{t=1}^{T} [aW_t + \sum_{i=1}^{G} B_i OT_{it} + \sum_{j=1}^{G} C_j OA_{jt} + \sum_{k=1}^{G} D_k HT_{kT} + \sum_{l=1}^{G} E_l HA_{lt} + \sum_{m=1}^{G} F_m PT_{mt} + \sum_{n=1}^{G} G_n PA_{nt} + \sum_{q=1}^{G} U_q VT_{qt} + \sum_{r=1}^{G} S_r VA_{rt}]$$

Subject to the constraints:

$$W_t - W_{t-1} - HT_t + HA_t = 0$$

$$P_t - P_{t-1} - PT_t + PT_t = 0$$

$$I_t - 320 - VT_t + VA_t = 0$$

$$P_t - 6W_t - OT_t + OA_t = 0$$

$$P_t - I_t + I_{t-1} = D_t$$

$$W_t \geq 0$$

$$P_t \geq 0$$

$$\sum_{i=1}^{G} b_i OT_{it} - OT_t + ZW_t = 0$$

$$\sum_{j=1}^{G} e_j OA_{jt} - OA_t + ZW_t = 0$$

$$\sum_{k=1}^{G} d_k HT_{kt} - HT_t = 0$$

$$\sum_{l=1}^{G} e_l HA_{lt} - HA_t = 0$$

$$\sum_{m=1}^{G} f_m PT_{mt} - PT_t = 0$$

$$\sum_{n=1}^{G} g_n PA_{nt} - PA_t = 0$$

$$\sum_{q=1}^{G} u_q VT_{qt} - VT_t + 320 = 0$$

$$\sum_{r=1}^{G} s_r VA_{rt} - VA_t + 320 = 0$$

for all $t = 1,2 \ldots T$. *Note:* G may vary for each variable depending on the desired quality of the fit of the linear segments to the cost structure.

Application of the separable programming model

The separable programming model was applied to two alternative cost structures:

- The first run was done using costs as given by the fourth-order cost functions developed by Goodman. Each cost function was approximated by six piecewise linear segments. Thus $G = 6$ in the formulation. This run will be called SEP (model application).
- The second run was done by using directly the hypothetical historical cost data shown in Table 1. Six piecewise linear segments were used to approximate each variable. This run will be called SEP (direct application).

The cost terms of both cost structures were graphed to determine the function values of the linear segments and the associated cost coefficients. It must be emphasized that the model developed when fitting the linear segments to the real world cost structure given in Table 1 must be

Table 1 Hypothetical-historical cost data by Goodman[1]

$\lvert W_t - W_{t-1}\rvert$	Cost	$\lvert P_t - P_{t-1}\rvert$	Cost	$\lvert I_t - I'\rvert$	Cost	$\lvert P_t - ZW_t\rvert$	Cost
0	0	1	1	1	1	0	0
1	66	2	2	2	2	1	1
2	1 001	4	24	3	9	2	4
3	5 210	5	68	4	28	3	14
4	20 100	7	225	6	122	5	131
5	38 120	10	1 049	8	392	7	457
7	86 300	16	6 310	11	1 370	10	1 876
9	139 200	22	26 100	15	5 417	12	3 780
12	224 400	34	123 400	21	18 240	14	7 795
14	279 600	52	487 200	39	231 200	18	20 600
19	401 100	87	1 140 000	51	474 400	22	34 900
25	698 700	150	2 224 000	70	702 500	30	58 200

Table 2 Aggregate production plan using separable programming applied to the high-order cost model and applied to the real-world cost structure

Period	Demand	Production (Units) Model application	Production (Units) Direct application	Work-force (Men) Model application	Work-force (Men) Direct application	Inventory (Units) Model application	Inventory (Units) Direct application
0		450	450	75	75	320	320
1	430	446	441	73	73	336	331
2	447	431	424	71	70	319	308
3	440	409	402	67	67	289	271
4	316	378	380	64	65	350	335
5	397	361	375	62	63	314	313
6	375	349	368	60	62	289	305
7	292	364	368	62	62	360	381
8	458	395	390	64	64	297	312
9	400	383	383	63	63	280	295
10	350	352	361	60	61	282	305
11	284	361	353	64	61	359	374
12	400	401	375	69	64	360	348
13	483	441	464	74	75	318	329
14	509	478	486	79	79	288	305
15	500	493	500	84	82	280	305
16	475	508	492	89	84	313	322
17	500	548	513	94	88	360	335
18	600	600	629	99	104	360	364
19	700	668	663	108	109	328	327
20	700	708	685	113	112	337	312
21	725	668	663	108	109	280	250
22	600	600	600	103	102	280	250
23	432	560	572	98	97	408	390
24	615	549	560	94	95	342	335

Total cost — (Calculated by using costs in Table 1))
Model application: R9 817 794
Direct application: R9 818 227

Table 3 Aggregate production plan using the sectioning search and goal programming models of Goodman

Period	Demand	Production (Units) Sectioning search	Production (Units) Goal programming	Work-force (Men) Sectioning search	Work-force (Men) Goal programming	Inventory (Units Sectioning search	Inventory (Units Goal programming
1	430	431	450	75	75	301	320
2	447	440	447	72	74	294	320
3	440	426	403	69	67	280	283
4	316	392	375	65	63	356	342
5	397	374	375	62	63	333	320
6	375	348	375	59	63	306	320
7	292	348	375	60	63	362	403
8	458	386	375	63	63	290	320
9	400	391	375	64	63	281	295
10	350	355	353	61	59	286	298
11	284	356	353	63	59	358	367
12	400	399	353	68	59	357	320
13	483	444	483	73	30	318	320
14	509	481	496	78	83	290	307
15	500	488	496	83	83	278	303
16	475	508	496	88	83	311	324
17	500	552	496	94	83	363	320
18	600	607	600	101	100	370	320
19	700	662	700	107	117	332	320
20	700	699	700	112	117	331	320
21	725	659	700	107	117	265	295
22	600	600	557	101	93	265	252
23	432	545	557	95	93	378	377
24	615	553	557	93	93	316	319

Total cost — (Calculated by using costs in Table 1)
Sectioning search: R10 625 200
Goal programming: R12 237 846

Table 4 A comparison of the cost areas and total cost associated with each solution methodology (All costs calculated using Table 1)

Costs	Sectioning search (last 13 periods) R	Goal programming (last 13 periods) R	Separable programming: Application of model costs (last 13 periods) R	Separable programming: Application of model costs (24 periods) R	Separable programming: Direct cost application (last 13 periods) R	Separable programming: Direct cost application (24 periods) R
Regular work-force	408 000 (100)[a]	408 340 (100)	412 080 (101)	653 480	408 000 (100)	649 740
Change in production rate	3 856 718 (100)	6 783 299 (176)	3 724 148 (97)	4 074 351	3 838 973 (99)	3 954 773
Changes in work-force	602 981 (100)	1 859 610 (308)	578 620 (96)	635 312	749 782 (124)	765 339
Overtime and idletime	179 565 (100)	46 (0)	270 616 (151)	320 716	44 941 (25)	53 406
Inventory and shortages	3 015 479 (100)	1 376 231 (46)	2 733 031 (91)	4 133 935	2 744 128 (91)	4 394 969
Total cost	8 062 743 (100)	10 427 520 (129)	7 718 495 (96)	9 817 794	7 785 824 (97)	9 818 227

[a]Numbers in brackets are percentages.

regarded as the better model — the fit to the Goodman model was done for control purposes.

Results and comparison

In order to compare the results of the two SEP models with that of the two approaches, goal programming and sectioning search used by Goodman, a twenty-four-period planning horizon was used. The production plans for both SEP models are given in Table 2 and can, for all practical purposes, be regarded as similar. The total cost difference is negligible. The comparison between the sectioning search model and the goal programming model for this planning period is given in Table 3. These results illustrated clearly that the cost structure used is very sensitive, so that small deviations from the global optimum plan involves large changes in the total cost. Costs are calculated for each of the production plans from the cost data in Table 1 using interpolation where necessary.

A comparison was made between the results of the four models mentioned above, on the basis of costs in the various cost areas. The starting conditions used by Goodman were unknown and therefore only the last 13 periods are compared to eliminate its effect. The comparison is given in Table 4. SEP (model application) offered an improvement of more than 4% in the total cost in comparison with the sectioning search model and performs 26% better than the GP models. These improvements are somewhat lower in the case of the SEP (direct application) — 3,4% and 25% respectively. A percentage-wise comparison is also made between the models in Table 4. The sectioning search results are used as a basis for this comparison. Major differences can be summarised as indicated in Table 5.

Table 5 A comparison between the various models, using sectioning search results as a basis

Model	Major differences compared to the sectioning search results
Goal programming	Changes in the work-force is dominant (approximately three times more), no overtime/undertime and less inventory. Higher total cost.
SEP (model application)	More overtime/undertime and less inventory. Total cost less.
SEP (direct application)	More changes in the work-force, less overtime/undertime and less inventory. Total cost less.

Conclusion

In conclusion the following advantages of the SEP approach must be underlined. This is a mathematical programming technique that can be used in a very flexible way. More linear segments may be added in cases where more accuracy is needed. In the case of monotonous cost increases (e.g. concave cost structures), the ordinary simplex algorithm may be used to solve the problem. The linear segments can be fitted graphically to real-world cost data without deriving, by means of laborious curve fitting methods, a complex mathematical model. In many cases ordinary linear curves, while in other instances (e.g. overtime pay as a linear function of normaltime pay) linear segments, represent the real world.

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